The Greenhouse Effect within an Analytic Model of the Atmosphere

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Within a simplified atmospheric model the greenhouse effect is treated by analytical methods starting from physical first principles. The influence of solar radiation, absorption cross sections of the greenhouse molecules, and cloud formation on the earth's temperature is shown and discussed explicitly by mathematical formulae in contrast to the climate simulations. The application of our analytical results on the production of $20 \cdot 10^9$ t of CO_2 per year yields an enlargement of the earth's surface temperature of $2.3 \cdot 10^{-2}$ °C per year in agreement with other estimations.

Key words: Global Properties of the Atmosphere; Influence of Greenhouse Gases and Clouds; Change of the Temperature.

1. Introduction

Usually the change of the climate of the earth is treated by numerical simulations with the aim to take into account all imaginable influences in order to get a detailed picture of the behaviour of the climate, e. g. in consequence of the production of greenhouse gases. But by this procedure the survey is lost. On the other hand, this is guaranteed, if one restricts oneself to an atmospheric model considering only the most important properties, which can be solved by analytical methods. This is the idea of the present paper, so that everybody with a sufficient knowledge in physics and in higher mathematics can understand qualitatively as well as quantitatively the behaviour of the atmosphere as a consequence of an enlargement of its content of greenhouse gases[1]. By such an analytical way the influence of solar radiation, absorption cross sections of the greenhouse molecules and cloud formation on the earth's temperature can be studied and discussed explicitly.

2. The Model

In view of the solar constant the model starts from a nearly constant mean energy flux J of the solar radiation on the surface of the earth; this radiation has short wavelengths ($\lambda_{\max} = 4.8 \cdot 10^{-5}$ cm) and reaches the surface more or less immediately. In consequence of the absorption of this radiation the earth's surface will be heated and radiates infrared rays with wavelengths around $1.7 \cdot 10^{-3}$ cm into the atmosphere. The mean temperature in the atmosphere may be T, its value at the surface T_0; the mean temperature T_E of the earth's surface itself will be determined later.

As long as the mean free path length l of the infrared photons is small compared with the thickness of the atmosphere as a consequence of absorption and reemission by the molecules of the greenhouse gases we have an energy transport by radiation in form of diffusion of the infrared photons connected with an energy flux density (1. Fick's law corresponding to the 2. law of thermodynamics):

$$\vec{j} = -\lambda \vec{\nabla} T \quad (\lambda > 0), \tag{1}$$

where

$$\lambda \sim l = \left(\sum_Q n_Q \sigma_Q \right)^{-1} \tag{2}$$

is the "photon conductivity" of the atmosphere; n_Q is the number density of the molecules of the greenhouse gases, σ_Q their effective absorption cross section for infrared photons, and Q indicates the different greenhouse gases. The absorption cross section σ_Q is determined in the first step by the quantum mechanical transition probabilities and is given in spectral decomposition by (sharp line of frequency $\nu_{Qnm} =$

[1] It may be historically interesting, that the influence of atmospheric absorbing molecules on the earth's temperature has been investigated already very early in 1827 by Fourier [1] and in 1838 by Pouillet [2] and especially in 1896 by Arrhenius [3]. But their results could be only preliminary because of the incomplete physical basis at that time.

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$|E_{Qm} - E_{Qn}|/h)$

$$\sigma_Q(\nu) = \frac{2\pi}{3}\frac{|E_{Qm} - E_{Qn}|}{\hbar^2 c}|\vec{d}_{Qmn}|^2\delta(\nu - \nu_{Qnm}) \quad (3)$$

with the dipole matrix element $\vec{d}_{Qmn}$ of the oscillating and rotating molecules. However, in (2) the effective absorption cross section of the greenhouse molecules is needed, i. e. (3) must be additionally multiplied with the probability $w(E_{Qm}, T)$, that the absorbing energy eigenstate E_{Qm} of the molecule is occupied. For the case of thermodynamic equilibrium (no saturation) w is given by the Boltzmann distribution. Furthermore we average in the following the spectral absorption cross section $\sigma_Q(\nu)$ with respect to the frequency range of the infrared radiation and get

$$\begin{aligned}\sigma_Q(T) &= \overline{\sigma_Q(\nu)}^{\nu} \\ &= \sum_{m,n}\int_0^\infty w_Q(E_{Qm}, T)\sigma_Q(\nu)I(\nu, T)\mathrm{d}\nu \quad (4) \\ &\qquad \Big/ \int_0^\infty I(\nu, T)\mathrm{d}\nu,\end{aligned}$$

where $I(\nu, T)$ represents the radiation spectrum for the temperature T, which we approximate by that of the black body radiation. However, also in that simple case we cannot calculate $\sigma_Q(T)$ "ab initio" according to (4). Therefore we approximate $\sigma_Q(T)$ within the relevant frequency range by the tangent of the real course in the double-logarithmic representation, i. e. by the power law

$$\sigma_Q(T) = \tilde{\sigma}_Q/T^\kappa, \quad \tilde{\sigma}_Q = \text{const.} \quad (5)$$

with $\kappa \leq 4$ in view of the fact, that the denominator in (4) is proportional to T^4 (Stefan-Boltzmann law); such an approximation is very reliable for a large temperature range. The exact value of the exponent κ will be determined later by fitting the results to the observation. In contrast to this the photon conductivity in (1) can be given exactly in the case of black body radiation and reads [4]

$$\lambda = \frac{16}{3}\sigma l T^3, \quad (6)$$

where $\sigma = \pi^2 k^4/60c^2\hbar^3 = 5.67 \cdot 10^{-8}\ \mathrm{W/m^2\,K^4}$ is the Stefan-Boltzmann-constant. The energy flux density (1) goes over into free radiation propagation without scattering, if the free path length l of the infrared photons is sufficiently large; then the energy flux J of the sun will be re-emitted into the universe. But before the balance equation is valid (radiation energy conservation),

$$J = \oint \vec{j}\mathrm{d}\vec{f} \quad (7)$$

for every closed surface around the earth.

According to this model the temperature distribution of the atmosphere is determined by the absorption of infrared radiation, whereby we suppose local thermodynamic equilibrium. Heat conductivity and convection are neglected in the first step, but heat conductivity could be taken into account very easily by an additional term in (6); convection will be treated subsequently in Section 5 and cloud formation in Section 6. On the other hand the day-night change, the summer-winter differences and the variations with respect to the geographic altitude as well as the influences of winds and oceanic streams are neglected completely. For the case, that the free path length of the infrared photons will be comparable with the thickness of the atmosphere, the model loses its applicability.

3. Temperature, Density and Pressure Distributions of the Atmosphere

With respect to the conservation of the radiation energy the integral (7) is valid for any closed surface around the earth, especially for any sphere of radius r. Herewith we find

$$\partial T/\partial r = T' = -\frac{3I\sum_Q \tilde{\sigma}_Q n_Q}{16\sigma r^2 T^{3+\kappa}}, \quad I = J/4\pi. \quad (8)$$

Additionally there exists hydrodynamical equilibrium in the atmosphere, i. e. the static Euler equation is valid (differential barometric equation):

$$\vec{\nabla}p + \rho\vec{\nabla}\phi = 0, \quad \phi = -\frac{MG}{r} \quad (9)$$

(without self-gravitation of the air),

where M is the mass of the earth, G the Newtonian gravitational constant. If

$$x_Q = n_Q/n_\mathrm{L} \quad (10)$$

is the ratio between the number density n_Q of the greenhouse molecules in question and that of the air molecules n_L, and

$$n = n_\mathrm{L} + \sum_Q n_Q \quad (11)$$

represents the total number density of molecules, then

$$n_\mathrm{L} = n\Big/\left(1+\sum_Q x_Q\right), \qquad n_Q = n x_Q\Big/\left(1+\sum_{Q'} x_{Q'}\right) \tag{12}$$

is valid and for the density ρ and pressure p of the atmosphere we find, under the assumption of an ideal gas for the atmosphere,

$$\rho = n\frac{m_\mathrm{L}+\sum_Q x_Q m_Q}{1+\sum_Q x_Q}, \quad p = nkT, \tag{13}$$

where k is the Boltzmann constant, m_Q the mass of the greenhouse molecules in question, and m_L the mean mass of the air molecules (mean molecular weight 28.8). The influence of the small radiation pressure is neglected. Insertion of (13) in (9) gives additionally to (8) a second differential equation

$$n'kT + nkT' + \frac{m_\mathrm{L}+\sum_Q x_Q m_Q}{1+\sum_Q x_Q} MG\frac{n}{r^2} = 0 \tag{14}$$

assuming spherical symmetry. Here the ratio x_Q [see (10)] is considered as a constant parameter in good agreement with the observations. From (8) and (14) both variables $n(r)$ and $T(r)$ are to be determined.

Solving (8) with respect to n/r^2 and inserting into (14) results in the exact differential equation

$$n'kT + nkT' - \frac{m_\mathrm{L}+\sum_Q x_Q m_Q}{\sum_Q x_Q \tilde{\sigma}_Q}\frac{16\sigma MG}{3I}T^{3+\kappa}T' = 0 \tag{15}$$

with the solution

$$n = \frac{a^2}{k}T^{3+\kappa} - A^2/kT, \tag{16}$$

where A^2 is the integration constant and a^2 is given by

$$a^2 = \frac{(m_\mathrm{L}+\sum_Q x_Q m_Q)}{\sum_Q x_Q \tilde{\sigma}_Q}\frac{4\sigma MG}{3(1+\kappa/4)I}. \tag{17}$$

From $a^2 > 0$ it follows $\kappa > -4$. The sign of the integration constant A^2 is chosen in such a way, that n is a monotonic function of T and that the atmosphere possesses a well defined outer border ($n = 0$, $\rho = 0$, $p = 0$) with the border temperature

$$T_\mathrm{G} = \left(\frac{A^2}{a^2}\right)^{1/(4+\kappa)}. \tag{18}$$

With the boundary condition $T_\mathrm{G} = 0$ it follows $A^2 = 0$.

Insertion of (16) in (8) or (14) results immediately in the following differential equation for $T(r)$:

$$(4+\kappa)kT' + \frac{m_\mathrm{L}+\sum_Q x_Q m_Q}{1+\sum_Q x_Q}\frac{MG}{r^2} = 0. \tag{19}$$

By separation of the variables we find with the integration constant B the solution

$$kT = \frac{m_\mathrm{L}+\sum_Q x_Q m_Q}{(4+\kappa)(1+\sum_Q x_Q)}\frac{MG}{r} - B. \tag{20}$$

Furthermore, we have the solution (16) in the form ($A^2 = 0$)

$$n = \frac{a^2}{k}T^{3+\kappa}, \quad T_\mathrm{G} = 0 \tag{21}$$

and, as border of the atmosphere ($T \to T_\mathrm{G} = 0$),

$$R = r_\mathrm{max} = b^2/B \tag{22}$$

with the gravitational coupling constant

$$b^2 = \frac{m_\mathrm{L}+\sum_Q x_Q m_Q}{(4+\kappa)(1+\sum_Q x_Q)}MG, \tag{23}$$

connected with a^2 [see (17)] according to

$$a^2 = \frac{1+\sum_Q x_Q}{\sum_Q x_Q \tilde{\sigma}_Q}b^2\frac{16\sigma}{3I}. \tag{24}$$

For $x_Q = 0$ we find $(1+\kappa/4)b^2 = 4.8\cdot 10^{-3}\ \mathrm{g\,cm^3\,s^{-2}}$ ($M = 6\cdot 10^{27}$ g), which is, in view of $x_Q \ll 1$, a very good approximation.

The remaining integration constant B in (20) will be determined finally by the total absorption cross section $\sum_Q \int n_Q \sigma_Q \mathrm{d}^3x$ of all greenhouse molecules. From (5), (12), (20) and (21) it follows:

$$\sum_Q \int n_Q \sigma_Q \mathrm{d}^3x = 4\pi b^2 \frac{16\sigma}{3k^4 I}\int_{R_0}^{R}\left(\frac{b^2}{r} - B\right)^3 r^2 \mathrm{d}r. \tag{25}$$

The calculation of the integral gives, after insertion of the upper limit according to (22),

$$\int_{R_0}^{R}\left(\frac{b^2}{r} - B\right)^3 r^2 \mathrm{d}r = b^6\left[\ln\frac{b^2}{BR_0} - \frac{11}{6}\right] + 3b^4 BR_0 - \frac{3}{2}b^2(BR_0)^2 + \frac{1}{3}(BR_0)^3. \tag{26}$$

Herewith equation (25) must be solved with respect to B or BR_0. But this is not exactly possible in view of the logarithmic term in (26). Therefore we make the ansatz

$$R = R_0(1+\varepsilon), \quad \varepsilon > 0, \tag{27}$$

according to which [see (22)]

$$BR_0 = b^2/(1+\varepsilon), \tag{28}$$

and consider $\varepsilon \ll 1$. Herewith we obtain from (26)

$$\int_{R_0}^{R} \left(\frac{b^2}{r} - B\right)^3 r^2 \mathrm{d}r = \frac{1}{4} b^6 \varepsilon^4 + O(\varepsilon^5), \tag{29}$$

and (25) results in

$$\varepsilon = \frac{k}{2b^2} \left(\frac{3I \sum_Q \int n_Q \sigma_Q \mathrm{d}^3 x}{\pi\sigma}\right)^{1/4}, \tag{30}$$

whereby also BR_0 [see (28) and (31)] is given. The condition $\varepsilon \ll 1$ means an upper limit for the total absorption cross section of the greenhouse molecules.

Now we are able to determine the radial behaviour of the atmosphere. From (20) and (28) it follows, with the use of (30),

$$BR_0 = b^2 \left[1 - \frac{k}{2b^2} \left(\frac{3I \sum_Q \int n_Q \sigma_Q \mathrm{d}^3 x}{\pi\sigma}\right)^{1/4}\right] \tag{31}$$

and[2]

$$kT = b^2 \left[\frac{1}{r} - \frac{1}{R_0} + \frac{1}{R_0} \frac{k}{2b^2} \left(\frac{3I \sum_Q \int n_Q \sigma_Q \mathrm{d}^3 x}{\pi\sigma}\right)^{1/4}\right] \tag{32}$$

with the temperature of the atmosphere at the earth's surface $T_0 = T\ (r = R_0)$:

$$T_0 = \frac{1}{2R_0} \left(\frac{3I \sum_Q \int n_Q \sigma_Q \mathrm{d}^3 x}{\pi\sigma}\right)^{1/4}. \tag{33}$$

Inversely (33) reads in view of (30) and (23)

$$\varepsilon = \frac{kT_0 R_0}{b^2} = \frac{(4+\kappa)(1+\sum_Q x_Q) kT_0 R_0}{(m_\mathrm{L} + \sum_Q x_Q m_Q) MG}. \tag{34}$$

[2]By the substitution $r = R_0 + h$ with $h \ll R_0$ it follows that T decreases linearly in first approximation with increasing h; one finds $\mathrm{d}T/\mathrm{d}h = -b^2/(kR_0^2) = \frac{-0.9}{1+\kappa/4}$ °C/100 m.

Because of $x_Q \ll 1$ it is possible to estimate the value of ε by (34); one finds, with $T_0 = 300$ K ($R_0 = 6370$ km),

$$\varepsilon = 5.5 \cdot 10^{-3}(1+\kappa/4) \quad (x_Q = 0), \tag{35}$$

so that the assumption $\varepsilon \ll 1$ is justified. With (32) and (21) the particle number density (density) and pressure are given according to (5) as

$$\begin{aligned} n &= \frac{1+\sum_Q x_Q}{\sum_Q x_Q \tilde{\sigma}_Q} b^2 \frac{16\sigma}{3kI} T^{3+\kappa}, \\ p &= \frac{1+\sum_Q x_Q}{\sum_Q x_Q \tilde{\sigma}_Q} b^2 \frac{16\sigma}{3I} T^{4+\kappa}, \end{aligned} \tag{36}$$

as well as the border of the atmosphere in view of (27) and (30) as

$$R = R_0 \left[1 + \frac{k}{2b^2} \left(\frac{3I \sum_Q \int n_Q \sigma_Q \mathrm{d}^3 x}{\pi\sigma}\right)^{1/4}\right] \tag{37}$$

and the thickness of the atmosphere as

$$H = R - R_0 = R_0 \varepsilon = \frac{kR_0}{2b^2} \left(\frac{3I \sum_Q \int n_Q \sigma_Q \mathrm{d}^3 x}{\pi\sigma}\right)^{1/4}. \tag{38}$$

With increasing values of n_Q (or x_Q) the atmosphere expands. Together with (35) we can estimate the thickness of the atmosphere as

$$H \simeq 35(1+\kappa/4)\ \mathrm{km}. \tag{39}$$

Accordingly the atmosphere would reach only the mesosphere because of $\kappa \leq 4$. The fact, that the atmosphere is actually higher, may depend on the additional heating of the upper atmosphere in consequence of solar ultraviolet absorption by O_3, which is neglected in our model.

Evidently, the features of the atmosphere are determined by the solar radiation I and the total absorption cross section $\sum_Q \int n_Q \sigma_Q \mathrm{d}^3 x$ of the greenhouse molecules as well as by the gravitational force of the earth ($b^2 \sim MG$). However in T_0 [see (33)] b^2 drops out(!), so that the atmospheric temperature at the surface is determined only by the product $I \sum_Q \int n_Q \sigma_Q \mathrm{d}^3 x$. The results (33) and (38) reflect very well the influence of the greenhouse molecules. In the case $n_Q \to 0$ the temperature T_0 and the thickness H go to zero. Simultaneously one finds by logarithmic differentiation of (33) immediately the enlargement ΔT_0 of the atmospheric temperature at the surface in consequence of an increasing $\Delta\int n_Q\sigma_Q \mathrm{d}^3x$ of the total absorption cross section of the greenhouse molecules

$$\frac{\Delta T_0}{T_0} = \frac{1}{4}\frac{\sum_Q \Delta\int n_Q\sigma_Q \mathrm{d}^3x}{\sum_Q \int n_Q\sigma_Q \mathrm{d}^3x} \tag{40}$$

as well as, in consequence of an increasing ΔI of the radiation power I of the sun,

$$\frac{\Delta T_0}{T_0} = \frac{1}{4}\frac{\Delta I}{I}. \tag{41}$$

4. The Temperature of the Surface of the Earth

The temperature T_E of the earth's surface is determined by the fact, that in the stationary case the surface must re-emit the infalling radiation power. This consists first of the radiation flux J of the sun and second of the infrared photons backscattered by the greenhouse molecules in the lower region of the atmosphere with the thickness of a mean free path length of the photons.

For calculation of the backscattered infrared radiation we have to determine at first the thickness $R_\mathrm{c} - R_0$ of the radiating region by the integral

$$\int_{R_0}^{R_\mathrm{c}} l^{-1}\mathrm{d}r = 1, \tag{42}$$

where l^{-1} is given according to (2), (5), (12), (21) and (32) by

$$l^{-1} = b^8\frac{16\sigma}{3k^4 I}\left(\frac{1}{r} - \frac{\beta}{R_0}\right)^3, \quad \beta = (1+\varepsilon)^{-1}. \tag{43}$$

Only solutions with $R_\mathrm{c} < R$ are useful; if they do not exist, the whole model is not applicable [c. f. (48)]. Performing the integral we get

$$b^8\frac{16\sigma}{3k^4 I}\left[\frac{1}{2}\left(\frac{1}{R_0^2} - \frac{1}{R_\mathrm{c}^2}\right) + 3\frac{\beta}{R_0}\left(\frac{1}{R_\mathrm{c}} - \frac{1}{R_0}\right) + 3\frac{\beta^2}{R_0^2}\ln\frac{R_\mathrm{c}}{R_0} - \frac{\beta^3}{R_0^3}(R_\mathrm{c} - R_0)\right] = 1. \tag{44}$$

This equation must be solved with respect to R_c, which is, however, impossible to be done exactly because of the logarithmic term. Therefore we make analogously to (27) the ansatz

$$R_\mathrm{c} = R_0(1+\delta), \quad \delta \ll 1 \quad (\delta > 0) \tag{45}$$

and expand equation (44) with respect to δ and ε. In this way we find

$$\delta\varepsilon^3 - \frac{3}{2}\delta^2\varepsilon^2 + \delta^3\varepsilon - \frac{1}{4}\delta^4 = \frac{3k^4 I R_0^2}{16\sigma b^8}. \tag{46}$$

This equation of fourth order in δ can be solved easily because of the binomial series on the left-hand side; the 4 roots are:

$$\begin{aligned} \delta_{1,2} &= \varepsilon \pm \left(\varepsilon^4 - \frac{3k^4 I R_0^2}{4\sigma b^8}\right)^{1/4}, \\ \delta_{3,4} &= \varepsilon \pm \mathrm{i}\left(\varepsilon^4 - \frac{3k^4 I R_0^2}{4\sigma b^8}\right)^{1/4}, \end{aligned} \tag{47}$$

from which, however, in view of $0 \le \delta \le \varepsilon$, only

$$\begin{aligned} \delta &= \varepsilon - \left(\varepsilon^4 - \frac{3k^4 I R_0^2}{4\sigma b^8}\right)^{1/4} \\ &= \varepsilon\left[1 - \left(1 - \frac{4\pi R_0^2}{\sum_Q\int n_Q\sigma_Q \mathrm{d}^3x}\right)^{1/4}\right] \end{aligned} \tag{48}$$

is useful. The atmosphere must be higher than the free path length of the infrared photons given by

$$\begin{aligned} R_\mathrm{c} - R_0 &= R_0\delta \\ &= \frac{kR_0}{2b^2}\left(\frac{3I\sum_Q\int n_Q\sigma_Q \mathrm{d}^3x}{\pi\sigma}\right)^{1/4} \\ &\cdot \left[1 - \left(1 - \frac{4\pi R_0^2}{\sum_Q\int n_Q\sigma_Q \mathrm{d}^3x}\right)^{1/4}\right]. \end{aligned} \tag{49}$$

Accordingly $\sum_Q\int n_Q\sigma_Q \mathrm{d}^3x \ge 4\pi R_0^2$ must be fulfilled for δ is real valued and $\delta \le \varepsilon$. On the other hand from $\varepsilon \ll 1$ it follows $\sum_Q\int n_Q\sigma_Q \mathrm{d}^3x \ll 16\pi\sigma b^8/(3k^4 I)$, which is, however, realized very well. The temperature of the atmosphere at $r = R_\mathrm{c}$ reads

$$\begin{aligned} T_\mathrm{c} &= \\ &\frac{1}{2R_0}\left[\frac{3I\sum_Q\int n_Q\sigma_Q \mathrm{d}^3x}{\pi\sigma}\left(1 - \frac{4\pi R_0^2}{\sum_Q\int n_Q\sigma_Q \mathrm{d}^3x}\right)\right]^{1/4} = \\ &T_0\left(1 - \frac{4\pi R_0^2}{\sum_Q\int n_Q\sigma_Q \mathrm{d}^3x}\right)^{1/4}. \end{aligned} \tag{50}$$

Now the backscattered radiation flux J_R will be calculated in such a way, that every greenhouse molecule in the lower region of the atmosphere [see (49)] radiates with its mean absorption cross section $\sigma_Q(T)$ [see (4)] as a black body with the atmospheric temperature $T(r)$ in direction to the earth's surface (Kirchhoff's law).[3] This gives, with respect to (2),

$$J_\mathrm{R} = 4\pi \int_{R_0}^{R_\mathrm{c}} \sigma T^4 r^2 \frac{\mathrm{d}r}{l}. \qquad (51)$$

Insertion of $T(r)$ and l according to (20), (23), (28) and (43) yields

$$\begin{aligned} J_\mathrm{R} &= 4\pi b^{16} \frac{16\sigma^2}{3k^8 I} \int_{R_0}^{R_\mathrm{c}} r^2 \left[\frac{1}{r} - \frac{\beta}{R_0}\right]^7 \mathrm{d}r \\ &= 4\pi b^{16} \frac{16\sigma^2}{3k^8 I} \left[\frac{1}{4}\left(\frac{1}{R_0^4} - \frac{1}{R_\mathrm{c}^4}\right)\right. \\ &\quad + \frac{7}{3}\frac{\beta}{R_0}\left(\frac{1}{R_\mathrm{c}^3} - \frac{1}{R_0^3}\right) - \frac{21}{2}\frac{\beta^2}{R_0^2}\left(\frac{1}{R_\mathrm{c}^2} - \frac{1}{R_0^2}\right) \\ &\quad + 35\frac{\beta^3}{R_0^3}\left(\frac{1}{R_\mathrm{c}} - \frac{1}{R_0}\right) + 35\frac{\beta^4}{R_0^4}\ln\frac{R_\mathrm{c}}{R_0} \\ &\quad - 21\frac{\beta^5}{R_0^5}(R_\mathrm{c} - R_0) + \frac{7}{2}\frac{\beta^6}{R_0^6}(R_\mathrm{c}^2 - R_0^2) \\ &\quad \left. - \frac{1}{3}\frac{\beta^7}{R_0^7}(R_\mathrm{c}^3 - R_0^3)\right]. \end{aligned} \qquad (52)$$

Because R_c is known only approximately [see (45) and (48)], it is necessary to expand also the right-hand side of (52) with respect to δ and ε [see (43) and (45)]. Considering only the leading terms we obtain

$$J_\mathrm{R} = 4\pi b^{16} \frac{2\sigma^2}{3k^8 I R_0^4} \varepsilon^8 \left[1 - \left(1 - \frac{\delta}{\varepsilon}\right)^8\right]. \qquad (53)$$

After insertion of ε and δ according to (30) and (48) we find the simple result

$$I_\mathrm{R} = J_\mathrm{R}/4\pi = \frac{3}{4} I \left[\frac{\sum_Q \int n_Q \sigma_Q \mathrm{d}^3x}{4\pi R_0^2} - \frac{1}{2}\right]. \qquad (54)$$

The energy balance for the determination of the surface temperature T_E of the earth reads now, under the assumption of black body radiation of the earth's surface,

$$\sigma R_0^2 T_\mathrm{E}^4 = I + I_\mathrm{R} \qquad (55)$$

and results after insertion of (54) in

$$T_\mathrm{E} = \left\{\frac{I}{\sigma R_0^2}\left[1 + \frac{3}{4}\left(\frac{\sum_Q \int n_Q \sigma_Q \mathrm{d}^3x}{4\pi R_0^2} - \frac{1}{2}\right)\right]\right\}^{1/4}. \qquad (56)$$

Obviously the 2. term within the bracket represents the greenhouse effect. Of course the limiting case $n_Q \to 0$ is not allowed because of $\delta \le \varepsilon$. However, we see from (55), that without backscattered infrared photons the surface temperature of the earth would be

$$T_\mathrm{E}(0) = T_\mathrm{E}(n_Q = 0) = \left(\frac{I}{\sigma R_0^2}\right)^{1/4}. \qquad (57)$$

Surprisingly the temperatures T_0, T_c and T_E are independent from the exponent κ of the power law (5) and independent from the gravitational force ($b^2 \sim MG$), but only determined by the energy flux I of the sun and by the influence of the greenhouse molecules $\int n_Q \sigma_Q \mathrm{d}^3x$, and increase with increasing n_Q similar to the thickness of the atmosphere (ε); however, the free path length (δ) decreases. Obviously, the presupposition of the model, that the thickness of the atmosphere must be larger than the free path length of the infrared photons, will be fulfilled better and better with increasing number of greenhouse molecules. On the other hand the thickness of the atmosphere and free path length of the infrared photons depend also on the exponent κ and increase with increasing values of κ by the factor $1 + \kappa/4$.

For the determination of T_0, T_c and T_E according to (33), (50) and (56) the knowledge of the value $\sum_Q \int n_Q \sigma_Q \mathrm{d}^3x$ is necessary. Because this value is unknown, we estimate it by the present temperature data. Without greenhouse molecules the mean temperature of the earth's surface would be, according to (57), $T_\mathrm{E}(0) = -18$ °C by the use of the solar constant.[4] However, the mean surface temperature of the earth amounts to $T_\mathrm{E} = +18$ °C. Herewith we find from (56) and (57)

$$\frac{\sum_Q \int n_Q \sigma_Q \mathrm{d}^3x}{4\pi R_0^2} = 1.43, \qquad (58)$$

and from (33) und (50) it follows

$$\begin{aligned} T_0 &= T_\mathrm{E}(0)\left(\frac{3}{4}\frac{\sum_Q \int n_Q \sigma_Q \mathrm{d}^3x}{4\pi R_0^2}\right)^{1/4} \\ &= -13.6\,^\circ\mathrm{C}; \qquad T_\mathrm{c} = -81\,^\circ\mathrm{C}. \end{aligned} \qquad (59)$$

[3] The right-hand side of (51) can be read also in such a way, that every stratum of thickness l within the region $R_\mathrm{c} \ge r \ge R_0$ radiates as a black body in direction of the earth.

[4] The primary solar constant amounts to 1.368 kW/m^2; subtraction of the albedo yields 957.6 W/m^2 at the earth's surface.

The magnitude of T_c is in good agreement with the temperature at the tropopause. Now the exact value of ε and herewith of the thickness H of the atmosphere can be determined from (34) and (38); one finds

$$\begin{aligned} \varepsilon &= 4.8 \cdot 10^{-3}(1+\kappa/4), \\ H &= 30.4(1+\kappa/4)\ \text{km} \end{aligned} \tag{60}$$

instead of the rough estimations (35) and (39), and from (48) and (49) we get

$$\begin{aligned} \delta &= 1.25 \cdot 10^{-3}(1+\kappa/4), \\ R_c - R_0 &= R_0\delta = 7.9(1+\kappa/4)\ \text{km}, \end{aligned} \tag{61}$$

where the last value is again in accordance with the height of the tropopause. However, the altitude H [see (60)] is too small in comparison with the observation, if we do not take into account the κ correction. Assuming a mean altitude of the atmosphere of 55 km, which corresponds to the stratopause, we obtain from (60)

$$\kappa = 3.2. \tag{62}$$

The height of the tropopause amounts then to 14.2 km. Because the free path length of the infrared photons reaches a height of 14 km – the height of the total atmosphere is 55 km – the model lies at the limit of validity. The temperature T_E of the earth's surface in dependence of $\sum_Q \int n_Q \sigma_Q \mathrm{d}^3x/4\pi R_0^2$ is shown in Figure 1.

For the relative change of the earth's surface temperature T_E in consequence of a small change of the number of the greenhouse molecules or of a small change of the solar radiation we find

$$\Delta T_E/T_E = \frac{3}{2}\left[5+6\frac{\sum_Q \int n_Q\sigma_Q \mathrm{d}^3x}{4\pi R_0^2}\right]^{-1}\frac{\sum_Q \Delta\int n_Q\sigma_Q \mathrm{d}^3x}{4\pi R_0^2}, \tag{63}$$

$$\Delta T_E/T_E = \frac{1}{4}\Delta I/I. \tag{64}$$

The solar radiation fluctuates at the earth's surface in the range of 0.3 W/m^2 during approximately 10 years in consequence of the activity of the sunspots; this gives according to (64) a temperature change of $2.3 \cdot 10^{-2}$ °C. On the other hand it follows from (63) together with (58) for the present situation

$$\begin{aligned} \frac{\Delta T_E}{T_E} &= 0.16\frac{\sum_Q \Delta\int n_Q\sigma_Q \mathrm{d}^3x}{\sum_Q \int n_Q\sigma_Q \mathrm{d}^3x} \\ &= 2.18 \cdot 10^{-20}\sum_Q \Delta\int n_Q\sigma_Q \mathrm{d}^3x \end{aligned} \tag{65}$$

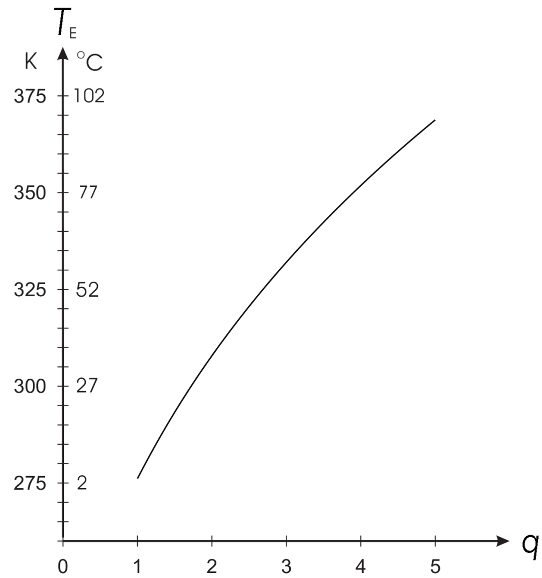

Fig. 1. Dependence of the temperature T_E of the earth's surface on the parameter $q = \sum_Q \int n_Q \sigma_Q \mathrm{d}^3x/4\pi R_0^2$ according to (56).

(σ_Q in cm^2). We state as result that changes of the intensity of the solar radiation give rise to temperature changes at the earth's surface by a factor 0.25 and changes of the absorption of the greenhouse molecules by a factor 0.16. In the latter cases the knowledge of the absorption cross sections is very essential. In order to calculate quantitatively the increase of the surface temperature of the earth in consequence of an increase of the concentration of the greenhouse gases, e. g. of the CO_2 concentration, the exact knowledge of the proper absorption cross section $\sigma_Q(T)$ is necessary according to (4) or (5).

A very rough estimation of $\sigma_Q(T_0)$ is possible by the air pressure $p_0 = p(T_0)$ at the earth's surface. From (36) it follows immediately

$$\sum_Q x_Q\sigma_Q(T_0) = \frac{(1+\sum_Q x_Q)}{p_0}b^2\frac{16\sigma}{3I}T_0^4. \tag{66}$$

Insertion of the known values of p_0, T_0, I and b^2 results for the case $\kappa = 3$ in ($x_Q \ll 1$)

$$\sum_Q x_Q\sigma_Q(T_0) = 3.78 \cdot 10^{-26}\ \text{cm}^2. \tag{67}$$

With $x_Q \simeq 3 \cdot 10^{-4} \hat{=} 0.03\%$ we find

$$\sum_Q \sigma_Q(T_0) \simeq 1.26 \cdot 10^{-22}\ \text{cm}^2. \tag{68}$$

If one distributes this total absorption cross section in very rough approximation equally on the four main greenhouse gases, an enlargement of $20 \cdot 10^9$ t of CO_2 per year corresponding to $\Delta N_{CO_2} = 2.7 \cdot 10^{38}$ leads according to (65) to an increase of the surface temperature

$$\frac{\Delta T_E}{T_E} = 2.78 \cdot 10^{-4} \Rightarrow \Delta T_E = 8.09 \cdot 10^{-2}\ ^\circ\mathrm{C} \quad (69)$$

per year ($N = \int n\, d^3x$).[5]

A more precise determination of ΔT_E in consequence of an increasing CO_2 concentration per year is possible by a half-empirical calculation of $\sigma_{CO_2}(T_0)$ by the integral (4). In view of the radiation temperature of the earth only one absorption line of CO_2 is important, namely that at the wavelength $1.5 \cdot 10^{-3}$ cm ($\mathrel{\hat{=}} \nu = 2 \cdot 10^{13}$ Hz); for this frequency no saturation exists in the atmosphere (for details see Appendix A). According to the "Hitran"-database [5] the corresponding absorption cross section amounts to $\sigma_{CO_2}(\nu) = 5 \cdot 10^{-18}\ \mathrm{cm}^2$. Assuming not a sharp line as in (3) but a (Doppler and impact) broadened line with a line width $\Delta\nu \simeq 10^8$ Hz we get from (4)

$$\sigma_{CO_2}(T_0) = 1.8 \cdot 10^{-23}\ \mathrm{cm}^2. \quad (70)$$

Correspondingly the temperature rise per year amounts now to

$$\Delta T_E = 4.6 \cdot 10^{-2}\ ^\circ\mathrm{C} \quad (71)$$

in consequence of the mentioned CO_2 production rate. However, the half of this rate is absorbed today by the oceans, so that the actual temperature rise lies at $\Delta T_E = 2.3 \cdot 10^{-2}$ °C. The weakness of any prediction of a temperature rise in consequence of the production of greenhouse gases is based on the fact that the absorption cross section $\sigma_Q(T)$ cannot be determined very exactly by the present observational data.

5. The Convection

The fact, that $T_E > T_0$ [cf. (33) and (56)], implies a convection in the lowest region of the earth's atmosphere, by which also a continuous temperature transition between the earth's surface and the atmosphere is established. Bubbles of atmospheric gas will be heated at the earth's surface to the temperature T_E and ascend within the cooler atmosphere under nearly adiabatic cooling until the surrounding atmospheric temperature T is reached. For the adiabatic cooling of the gas bubbles of the volume V

$$T_g V^{2/3} = \text{const.} \quad (72)$$

is valid, where T_g is the temperature of the gas bubble. For the changing volume

$$V = N k T_g / p, \quad (73)$$

where N is the molecule number in the bubble, is valid according to the ideal gas equation, where p is the pressure in the atmosphere equal to the pressure in the gas bubble, for which we find according to (36)

$$p = \text{const.}\ T^{4+\kappa}. \quad (74)$$

Insertion of (73) and (74) into (72) yields

$$T_g^{5/3} = \text{const.}\ T^{(8+2\kappa)/3}. \quad (75)$$

The constant in (75) will be determined at the earth's surface, where $T_g = T_E$ and $T = T_0$ is valid; thus it follows

$$T_g^{5/3} = T_E^{5/3} \left(\frac{T}{T_0}\right)^{(8+2\kappa)/3}. \quad (76)$$

The ascent of the gas bubbles, i. e. the convection, is stopped, when $T_g = T = T_K$ is reached; that means

$$T_K^{3+2\kappa} = T_0^{8+2\kappa} / T_E^5. \quad (77)$$

With the value of T_0 and T_E and with $\kappa = 3$ we obtain

$$T_K = 243.3\ \mathrm{K} \mathrel{\hat{=}} -29.7\ ^\circ\mathrm{C}. \quad (78)$$

This temperature corresponds according to the temperature behaviour (32) to the hight $[r(T_K) = R_K]$:

$$R_K - R_0 = 3.3\ \mathrm{km}. \quad (79)$$

Up to this altitude, which is much lower than the tropopause, convection is active in good agreement with the observation (weather). This result shows however simultaneously that convection is not important for the energy transport into the higher atmosphere. Therefore, the neglection of the convective energy transport within the model may be justified retrospectively.

[5] It can be shown for $\kappa = 3$, that $\Delta \int n_{CO_2} \sigma_{CO_2} d^3x \simeq 1.5 \Delta N_{CO_2} \sigma_{CO_2}(T_0)$. By this calculation one can also show, that $N_L = \frac{4}{7} \sum_Q \int n_Q \sigma_Q d^3x / \sum_Q x_Q \sigma_Q(T_0)$ and $p_0 = (m_L + \sum_Q x_Q m_Q) \frac{MG}{4\pi R_0^4} N_L$. In consequence of this we find $N_L = 1.1 \cdot 10^{44} \mathrel{\hat{=}} M_L = 5.2 \cdot 10^{21}$ g in very good agreement with the observation. The total number of the CO_2 molecules (0.038%) amounts to $4.2 \cdot 10^{40}$. The detailed calculation is presented in Appendix B.

6. The Influence of Cloud Formation

Clouds have a double influence on the temperature T_E of the earth's surface. First they reduce the energy flux I of the sun on the surface, and second they act similar to the greenhouse gases on the infrared radiation of the earth's surface. Both effects act against each other with respect to the temperature T_E.

The reduction of the energy flux of the sun follows from the differential Lambert law

$$dI = -I\frac{ds}{l}, \tag{80}$$

where $l^{-1} = \sigma_w n_w$ represents the free path length of the solar radiation in the clouds (σ_w absorption cross section of the water drops, n_w their number density) and ds means the infinitesimal distance in the cloud. The integral of (80) reads assuming nearly constant values for σ_w (Mie scattering) and n_w (stratification)

$$I = I_0 e^{-s/l}, \tag{81}$$

where I_0 is the primordial energy flux. Herewith it follows for the reduction of the intensity

$$\Delta I = I - I_0 = -I_0(1 - e^{-s/l}), \tag{82}$$

where s/l can be represented by

$$\frac{s}{l} = \frac{\sigma_w N_w}{4\pi R_0^2} \quad \left(N_w = \int n_w dx^3\right) \tag{83}$$

and is a measure for the covering of the earth's surface by clouds. Furthermore we consider only small changes of the cloudiness ($\Delta N_w \sigma_w \ll 4\pi R_0^2$). Then we can expand (82) with respect to ΔN_w and find together with (83)

$$\frac{\Delta I}{I} = -\frac{\sigma_w \Delta N_w}{4\pi R_0^2}. \tag{84}$$

Now, if we repeat the procedure of (55) considering additionally the cloud formation connected with reduction of the solar intensity and backscattering of the infrared photons at the temperature T_0 (stratification), we find, instead of (56),

$$T_E(N_w) = \left\{ \frac{I\left(1 + \frac{\Delta I}{I}\right)}{\sigma R_0^2} \left[\frac{5}{8} + \frac{3}{4} \cdot \frac{\sum_Q \int n_Q \sigma_Q dx^3}{4\pi R_0^2} \left(1 + \frac{\sigma_w \Delta N_w}{4\pi R_0^2}\right)\right]\right\}^{1/4}, \tag{85}$$

where $\Delta I/I$ is connected with $\sigma_w \Delta N_w$ according to (84). Because of the smallness of ΔI and ΔN_w we can expand and obtain finally

$$\frac{\Delta T_E}{T_E} = \frac{5}{4}\left[5 + 6\frac{\sum_Q \int n_Q \sigma_Q d^3x}{4\pi R_0^2}\right]^{-1} \frac{\Delta I}{I} \tag{86}$$

in consequence of the change of the cloudiness, where its reason is not important. Even cosmic rays are imaginable as cause [6]. With the present value for $\sum_Q \int n_Q \sigma_Q d^3x / 4\pi R_0^2$ [see (58)] it follows

$$\frac{\Delta T_E}{T_E} = 9.2 \cdot 10^{-2} \frac{\Delta I}{I}. \tag{87}$$

Formation of clouds means $\Delta I < 0$ [see (82)] and therefore the temperature T_E of the earth's surface decreases in view of (87) and vice versa. Unfortunately exact data about changes of the global cloudiness are not available.

In the case, that we have formation of clouds and enlargement of the concentration of greenhouse gases, we must add the results (63) and (86):

$$\left(\frac{\Delta T_E}{T_E}\right)_{tot} = \frac{3}{2}\left[5 + 6\frac{\sum_Q \int n_Q \sigma_Q d^3x}{4\pi R_0^2}\right]^{-1} \cdot \left[\frac{\sum_Q \Delta \int n_Q \sigma_Q d^3x}{4\pi R_0^2} - \frac{5}{6}\frac{\sigma_w \Delta N_w}{4\pi R_0^2}\right]. \tag{88}$$

Heating (1. term) and cooling (2. term) of the earth's surface compensate each other, if

$$\sigma_w \Delta N_w = \frac{6}{5} \sum_Q \Delta \int n_Q \sigma_Q d^3x \tag{89}$$

is valid. This relation does not seem to be unrealistic. But in order to decide this exactly, a backreaction mechanism is necessary to describe the coupling between cloud formation and increasing of the concentration of greenhouse gases, which requires also a better understanding of cloud formations. Such research will be started in 2010 at CERN [7]. There will be studied cloud formation more in detail under laboratory conditions in a cloud chamber, because also cosmic rays may have an influence on cloud formation.

7. Final Remarks

The analytical method presented here has the advantage, that the influence of the solar radiation, of the absorption cross sections of the greenhouse gases, and of the cloud formation on the earth's temperature are given by mathematical formulae explicitly and can be calculated quantitatively at any time. We are aware, however, that a critical assumption of our analytical considerations is that of local thermodynamic equilibrium of the atmosphere and the infrared radiation. This may be fulfilled today only approximately because of the actually small value of (58). Nevertheless, in spite of this, our results of increasing temperatures on the earth are in good agreement with numerical simulations, e. g. with the IPCC reports [8] as well as with other simple atmospheric models based on energy balance considerations [9 – 11]. However, for more reliable predictions of the evolution of the earth's temperature more precise determinations of the effective absorption cross sections of the greenhouse molecules and of the mechanism of cloud formation seem to be necessary.

Appendix A

The saturation condition can be derived very easily with the use of a two-energy-level system (see Fig. 2). In the stationary case it is valid for the occupation numbers N_1, N_2 of the two energy levels E_1 and E_2 ($E_2 > E_1$):

$$N_1 B_{12} u(\nu_{12}) = N_2 A_{21} + N_2 B_{21} u(\nu_{12}). \tag{A.1}$$

Here A_{21}, B_{21}, B_{12} are Einstein's transition probabilities for spontaneous and induced emission and absorption and $u(\nu_{12})$ is the spectral radiation energy density of the frequency $\nu_{12} = (E_2 - E_1)/h$. According to the quantum theory,

$$B_{12} = B_{21}, \quad A_{21} = 2u_0(\nu_{12}) B_{21}, \tag{A.2}$$

where $u_0(\nu_{12})$ is the spectral zero-point energy density of the radiation field. Insertion of (A.2) into (A.1) results in

$$2u_0(\nu_{12}) = \left(\frac{N_1}{N_2} - 1\right) u(\nu_{12}). \tag{A.3}$$

The saturation condition now reads

$$u(\nu_{12}) \gg 2u_0(\nu_{12}) \Rightarrow N_2 \to N_1. \tag{A.4}$$

In this case effectively no absorption happens: The spontaneous emission does not play a role and the induced emission compensates the absorption completely; the atmosphere becomes transparent. In view of the infrared radiation of the earth we choose now for $u(\nu_{12})$ thermal radiation; then

$$u(\nu_{12}) = \frac{8\pi}{c^3} \frac{h\nu_{12}^3}{e^{\frac{h\nu_{12}}{kT}} - 1}, \quad u_0(\nu_{12}) = \frac{4\pi}{c^3} h\nu_{12}^3 \tag{A.5}$$

is valid (Planck's formula). Herewith the saturation condition (A.4) takes the form

$$e^{\frac{h\nu_{12}}{kT}} - 1 \ll 1. \tag{A.6}$$

This condition is for $\nu_{12} = 2 \cdot 10^{13}$ Hz and $T \lesssim 300$ K not fulfilled for a large extent. Thus there exists no absorption saturation for CO_2 in the atmosphere.

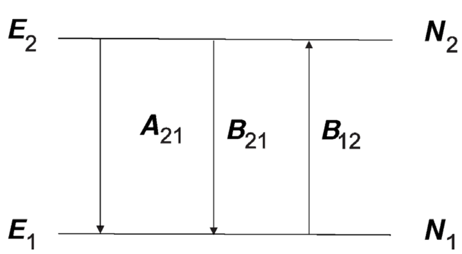

Fig. 2. Transitions in a 2-energy-level system according to absorption (B_{12}) and spontaneous (A_{21}) and induced (B_{21}) emission.

Appendix B

For the estimation of the change of the earth's surface temperature according to (65) the following problem arises: Empirically known is the production e. g. of CO_2 molecules per year, i. e. ΔN_{CO_2}. Thus we need the connection between ΔN_{CO_2} and $\Delta \int n_{CO_2} \sigma_{CO_2} d^3x$, which is used in (65).

For this we determine in a first step N_Q generally. According to (12) and (21) we find

$$N_Q = \frac{x_Q}{1 + \sum_{Q'} x_{Q'}} \frac{a^2}{k} \int T^{3+\kappa} d^3x. \tag{B.1}$$

Using (31) and (32) for substituting the temperature T we obtain

$$N_Q = 4\pi \frac{x_Q}{1 + \sum_{Q'} x_{Q'}} \cdot \int_{R_0}^{R} \frac{a^2}{k^{4+\kappa}} b^{2(3+\kappa)} \left(\frac{1}{r} - \frac{1}{R}\right)^{3+\kappa} r^2 dr. \tag{B.2}$$

For evaluating this integral we choose $\kappa = 3$ [see (62)] and use for R the relation (27) with $\varepsilon \ll 1$. Then we obtain

$$N_Q = \frac{4\pi}{7}\frac{x_Q}{1+\sum_{Q'} x_{Q'}}\frac{a^2 b^{12}}{k^7 R_0^3}\varepsilon^7 \\ = \frac{4}{7}\frac{x_Q}{\sum_{Q'} x_{Q'}\sigma_{Q'}(T_0)}\sum_{Q'}\int n_{Q'}\sigma_{Q'}\mathrm{d}^3x \tag{B.3}$$

after insertion of a^2 and ε according to (24) and (30), respectively. In view of (12),

$$N_\mathrm{L} = \frac{4}{7}\sum_Q \int n_Q\sigma_Q \mathrm{d}^3x \Big/ \sum_Q x_Q\sigma_Q(T_0) \tag{B.4}$$

follows immediately. A further useful relation follows from (36) for the atmospheric pressure at the earth's surface [cf. (66)]:

$$p_0 = \frac{1+\sum_Q x_Q}{\sum_Q x_Q\sigma_Q(T_0)} b^2 \frac{16\sigma}{3I} T_0^4. \tag{B.5}$$

After insertion of b^2 and T_0 according to (23) and (33), respectively, we find with respect to (B.4)

$$p_0 = \left(m_\mathrm{L} + \sum_Q x_Q m_Q\right)\frac{MG}{4\pi R_0^4} N_\mathrm{L}. \tag{B.6}$$

This formula is very interesting. It allows to calculate N_L ($x_Q \ll 1$; m_L, M, R_0, p_0 are known); then the value of $\sum_Q x_Q\sigma_Q(T_0)$ in view of (58) follows from (B.4). This will be used later.

In a second step we determine the integral $\int n_Q\sigma_Q\mathrm{d}^3x$. After insertion of (5), (12) and (21), we find

$$\int n_Q\sigma_Q \mathrm{d}^3x = 4\pi\frac{x_Q\tilde{\sigma}_Q}{1+\sum_{Q'} x_{Q'}}\frac{a^2}{k}\int_{R_0}^{R} T^3 r^2 \mathrm{d}r. \tag{B.7}$$

Substitution of T according to (31) and (32) results in

$$\int n_Q\sigma_Q \mathrm{d}^3x = 4\pi\frac{x_Q\tilde{\sigma}_Q}{1+\sum_{Q'} x_{Q'}}\frac{a^2}{k^4}\int_{R_0}^{R}\left(\frac{b^2}{r} - B\right)^3 r^2 \mathrm{d}r. \tag{B.8}$$

This integral is known from (29) and possesses the value $\frac{1}{4}b^6\varepsilon^4$ ($\varepsilon \ll 1$). Insertion of a^2 and ε according to (24) and (30) yields the interesting result, with the use of (5),

$$\int n_Q\sigma_Q \mathrm{d}^3x = \frac{x_Q\sigma_Q(T_0)}{\sum_{Q'} x_{Q'}\sigma_{Q'}(T_0)}\sum_{Q'}\int n_{Q'}\sigma_{Q'}\mathrm{d}^3x. \tag{B.9}$$

This relation can be proved immediately by insertion of n_Q and σ_Q.

Finally we combine (B.3) and (B.9) and get

$$\int n_Q\sigma_Q \mathrm{d}^3x = \frac{7}{4}N_Q\sigma_Q(T_0). \tag{B.10}$$

By this relation the desired connection between ΔN_Q and $\Delta\int n_Q\sigma_Q\mathrm{d}^3x$ can be deduced.

For small (infinitesimal) changes we find from (B.10)

$$\Delta\int n_Q\sigma_Q \mathrm{d}^3x = \frac{7}{4}[\sigma_Q(T_0)\Delta N_Q + N_Q\Delta\sigma_Q(T_0)], \tag{B.11}$$

where in the last term, according to (5) ($\kappa = 3$),

$$\Delta\sigma_Q(T_0) = -3\sigma_Q(T_0)\frac{\Delta T_0}{T_0} \tag{B.12}$$

is valid. By logarithmic differentiation of (33) one gets

$$\frac{\Delta T_0}{T_0} = \frac{1}{4}\frac{\Delta\sum_Q\int n_Q\sigma_Q\mathrm{d}^3x}{\sum_Q\int n_Q\sigma_Q\mathrm{d}^3x}, \tag{B.13}$$

and insertion of (B.12) and (B.13) into (B.11) yields

$$\Delta\int n_Q\sigma_Q\mathrm{d}^3x\left[1+\frac{21}{16}N_Q\sigma_Q(T_0)\Big/\sum_{Q'}\int n_{Q'}\sigma_{Q'}\mathrm{d}^3x\right] \\ = \frac{7}{4}\Delta N_Q\sigma_Q(T_0). \tag{B.14}$$

Finally we eliminate N_Q in the bracket by (B.3) and obtain

$$\Delta\int n_Q\sigma_Q\mathrm{d}^3x = \frac{\frac{7}{4}\Delta N_Q\sigma_Q(T_0)}{1+\frac{3}{4}\frac{x_Q\sigma_Q(T_0)}{\sum_{Q'} x_{Q'}\sigma_{Q'}(T_0)}}. \tag{B.15}$$

Herewith we have achieved our aim. If there would exist only one greenhouse gas ($Q = 1$ only) it follows $\Delta\int n_1\sigma_1\mathrm{d}^3x = \Delta N_1\sigma_1(T_0)$.

Now we apply the result (B.15) to the CO_2 problem. With an air pressure of 1 bar at the earth's surface we get from (B.6) $N_\mathrm{L} = 1.1\cdot 10^{44}$ and herewith, from (B.4) with the use of (58) [cf. (67)],

$$\sum_Q x_Q\sigma_Q(T_0) = 3.78\cdot 10^{-26}\ \mathrm{cm}^2. \tag{B.16}$$

On the other hand for CO_2 it is valid $x_{CO_2} = 3.8\cdot 10^{-4}$ and $\sigma_{CO_2}(T_0) = 1.8\cdot 10^{-23}\ \mathrm{cm}^2$ [see (70)]; then it follows

$$x_{CO_2}\sigma_{CO_2}(T_0) = 6.84\cdot 10^{-27}\ \mathrm{cm}^2. \tag{B.17}$$

Insertion of (B.16) and (B.17) into (B.15) yields for CO_2 the final result

$$\Delta \int n_{\mathrm{CO}_2} \sigma_{\mathrm{CO}_2} \mathrm{d}^3 x = 1.54 \cdot \Delta N_{\mathrm{CO}_2} \sigma_{\mathrm{CO}_2}(T_0). \quad \text{(B.18)}$$

With the approximative value (68) instead of (70) one obtains nearly the same result.

[1] J. B. Fourier, Mem. Acad. R. Sci. Inst. Fr. **VII** (1827).
[2] C. S. Pouillet, C. R. **VII**, 41 (1838).
[3] S. Arrhenius, Philos. Mag. J. Sci., 5. Series, **41**, 237 (1896).
[4] See e. g. H. Scheffler and H. Elsässer, Physik der Sterne und der Sonne, 2nd ed., Wissenschaftsverlag, Mannheim/Wien/Zürich 1990, p. 296.
[5] Hitran-database, http://cfa-www.harvard.edu/hitran /vibrational.html
[6] H. Svensmark and E. Friis-Christensen, Atmos. Solar.-Terrest. Phys. **59**, 1225 (1997); S. Rahmstorf, D. Archer, D. S. Ebel, O. Eugster, J. Jouzel, D. Maraun, G. A. Schmidt, J. Severinghaus, A. J. Weaver, and J. Zachos, Eos **85**, 38 (2004); N. J. Shaviv, Phys. Rev. Lett. **89**, 051102 (2002).
[7] J. Kirkby, Surv. Geophys. **28**, 333 (2008).
[8] IPCC Fourth Assessment Report, Climate Change 2007, http://www.ipcc.ch/ipccreports/ar4-syr.htm
[9] See e. g. D. L. Hartmann, Global Physical Climatology, Academic Press, San Diego, USA 1994.
[10] K. E. Trenberth (Ed.), Climate System Modeling, Cambridge University Press, New York 1992.
[11] K. McGuffie and A. Henderson-Sellers, A Climate Modeling Primer, Wiley, Chichester, UK 2005.